

COMPACT PERIODS OF EISENSTEIN SERIES OF ORTHOGONAL GROUPS OF RANK ONE AT EVEN PRIMES

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ABSTRACT. Fix a number field k with its adèle ring \mathbb{A} . Let $G = \mathrm{O}(n+3)$ be an orthogonal group of k -rank 1 and $H = \mathrm{O}(n+2)$ a k -anisotropic subgroup. We have previously described how to factor the global period

$$(E_\varphi, F)_H = \int_{H_k \backslash H_{\mathbb{A}}} E_\varphi \cdot \overline{F}$$

of a spherical Eisenstein series E_φ of G against a cuspform F of H into an Euler product. Here, we describe how to evaluate the factors at even primes. When the local field is unramified, we carry out the computation in all cases. We show also concrete examples of the complete period when $k = \mathbb{Q}$. The results are consistent with the Gross–Prasad conjecture.

INTRODUCTION

Fix a number field k ; in some examples to follow, we take $k = \mathbb{Q}$. Equip k^{n+3} with a quadratic form $\langle \cdot, \cdot \rangle$ with matrix

$$\begin{pmatrix} 1 & & \\ & * & \\ & & -1 \end{pmatrix}$$

with respect to the orthogonal decomposition $k^{n+3} = (k \cdot e_+) \oplus k^{n+1} \oplus (k \cdot e_-)$. Let $G = \mathrm{O}(n+3)$ and its subgroups act always on the right. Let $H = \mathrm{O}(n+2)$ be the fixer of e_- and $\Theta = \mathrm{O}(n+1)$ be the fixer of both e_+ and e_- . We consider only the case when k^{n+2} is anisotropic; in particular, G has k -rank 1 and H is k -anisotropic.

In this paper, we compute some automorphic periods associated to G and H . Such periods contain information about representations of those groups, as well as information of interest about certain L -functions. The Gross–Prasad conjecture [10–12] predicts that a representation of $\mathrm{O}(n)$ occurs in a representation of $\mathrm{O}(n+1)$ if and only if the corresponding tensor product L -function is nonzero on $\mathrm{Re} s = \frac{1}{2}$. The results we obtain are consistent with the prediction.

Because G is a reductive group, we can use its parabolics to organize the spectral decomposition of functions in $L^2(G_k \backslash G_{\mathbb{A}})$. In our case, there is only one parabolic up to conjugacy; in section 1, we choose a representative P , with Levi component of the form $M \cong \Theta \times \mathrm{GL}(1)$ and unipotent radical N . Let also K^G be some maximal compact; we will only consider right K^G -invariant functions, so-called *spherical*

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functions. We have two main families of spectral components (with more flags of parabolics we would have more).

The *cuspidal components* decompose discretely, and are in some sense the analogue of the compact group components we have in H ; we will not be much concerned with them in this paper.

The *Eisenstein series* are indexed by cuspidal components η along Θ and by characters ω along $\mathrm{GL}(1)$. These so-called *Hecke characters* are the analogue of Dirichlet characters in number fields other than \mathbb{Q} .

We use θ to indicate an element of $\Theta \subset G$ and m_λ for the element in G corresponding to $\lambda \in \mathrm{GL}(1)$. Given such η and ω , we can extend

$$\varphi(m_\lambda \theta) = \omega(m_\lambda) \cdot \eta(\theta)$$

by left $N_{\mathbb{A}}$ -invariance and right K^G -invariance. We will sometimes use $\varphi_{\omega, \eta}$; other times φ_ω (when $\eta = 1$); other times simply φ . We define the Eisenstein series $E_\varphi = E_{\omega, \eta}$ as the meromorphic continuation of

$$\sum_{\gamma \in P_k \backslash G_k} \varphi(\gamma g).$$

The characters ω are indexed in particular (but in number fields other than $k = \mathbb{Q}$, not only) by a continuous parameter s chosen along $\mathrm{Re} s = \frac{1}{2}$ and appearing in the form δ_P^s , where δ_P is the modular function of the parabolic P and $\frac{1}{2}$ has to do with normalizations of Haar measures.

Finally, the sum defining an Eisenstein series converges only for sufficiently large $\mathrm{Re} s$. Therefore, we must also include any residues to the right of $\mathrm{Re} s = \frac{1}{2}$ in the spectral decomposition.

Let $E_\varphi = E_{\omega, \eta}$ be the spherical Eisenstein series associated with the Hecke character $\omega : \mathrm{GL}(1) \rightarrow \mathbb{C}^\times$ and the cuspidal component η on Θ , and let F be cuspidal on H . We are interested in the period

$$(E_\varphi, F)_H = \int_{H_k \backslash H_{\mathbb{A}}} E_\varphi \cdot \overline{F}.$$

(Maybe some aspects of this computation may guide the corresponding computation for periods along H of cuspidal components of G , but the attempt must be left for another occasion.)

These same periods (called there global Shintani functions) were used by Katu, Murase, and Sugano [16, 22] to obtain and study integral expressions for standard L -functions of the orthogonal group. We already mentioned the Gross–Prasad conjecture. Ichino and Ikeda [14] discuss further details and broader context is provided in papers by Gross, Reeder [13], Jacquet, Lapid, Offen, and/or Rogawski [17, 19, 20], Jiang [18], and Sakellaridis and Venkatesh [25, 26].

Both the uncorrected global period and all correction factors computed so far are indeed nonzero.

Using the Phragmén–Lindelöf principle, it is often possible to obtain *convex* bounds for asymptotics of moments of automorphic L -functions. The Lindelöf hypothesis (a consequence of the Riemann hypothesis) yields significantly better bounds, but any *subconvex* bounds are of interest. Iwaniec and Sarnak [15] survey important ideas about L -functions, including subconvexity problems.

Diaconu and Garrett [5, 6] used a specific spectral identity to obtain subconvex bounds for second moments of automorphic forms in $\mathrm{GL}(2)$ over *any* number field

k . That strategy has been explored in other other papers by them and/or Goldfeld [6–8] and used by Letang [21]. In another paper [3] (from which, incidentally, this paragraph and the third before it are taken almost verbatim), this author has applied that strategy to the periods discussed here to obtain a spectral identity for second moments of Eisenstein series of G .

Elsewhere [2], the author has discussed how to factor the period

$$(E_\varphi, F)_H = \int_{H_k \backslash H_\mathbb{A}} E_\varphi \cdot \overline{F}$$

into an Euler product and how to compute its local factors at odd primes. For the reader's convenience, we recapitulate those results as briefly as possible.

Because $H \supset \Theta$ is a Gelfand pair [1, 16] and η is spherical, we have

$$\int_{\Theta_k \backslash \Theta_\mathbb{A}} \eta(\theta) \cdot \overline{F}(\theta h) d\theta = (\eta, F)_\Theta \cdot f(h),$$

where f is a spherical vector of $\text{Ind}_\Theta^H 1$ normalized by $f(1) = 1$. Let $\pi = \otimes_v \pi_v$ be the irreducible representation generated by f and $\omega = \otimes_v \omega_v$. Letting f_v be a generator of π_v normalized by $f_v(1) = 1$, the global period is

$$(1) \quad (E_\varphi, F)_H = (\eta, F)_\Theta \cdot \int_{\Theta_\mathbb{A} \backslash H_\mathbb{A}} \varphi_\omega \cdot f = (\eta, F)_\Theta \cdot C_f \cdot \prod_v \int_{\Theta_v \backslash H_v} \varphi_{\omega, v} \cdot f_v,$$

for some constant C_f (which is 1 when $F = 1$).

Because $\varphi_{\omega, v}$ and f_v are spherical, and $\varphi_v(1) = f_v(1) = 1$, the local integral is simply $\text{vol}(\Theta_v \backslash H_v)$ at anisotropic places v .

At isotropic places, we consider some parabolic $Q_v \subset H_v$. If the period is nonzero, then π_v is a quotient of a degenerate unramified principal series with respect to the Levi component of Q_v and with parameter β_v .

Let Δ be the discriminant of the restriction of $\langle \cdot, \cdot \rangle$ to k^{n+2} . In the preceding paper [2], we determined the local factors at odd primes. In this paper, we discuss what happens at even primes.

In section 1, we describe in more detail the conventions that we used at odd primes and will adapt to the even primes; in particular, we introduce the function X and show how the local period may be readily obtained from it.

In section 2, we explain what the required adaptations are and articulate a general method to determine the function X at an even prime, based on the number of solutions of an equation on finitely many finite rings of characteristic 2.

In section 3, we discuss two methods to count such solutions. The more flexible of them, however, is only applicable when the local fields is unramified.

In the subsequent sections, we apply either of those methods to each of the anisotropic forms, thus obtaining the function X associated to that form. (When we apply the second method, the computation is limited to the unramified case.)

In all computations, it will transpire that only the dimension of the anisotropic component, the Hasse–Minkowski invariant, whether the discriminant is a unit, and (when it is a unit) its quadratic defect, play a role in the outcome.

To obtain the complete periods, we would need to know the factors at all places, including the ramified cases not established here and the archimedean places where the form is isotropic. But if we restrict ourselves to $k = \mathbb{Q}$, we have no ramification at the even prime and we can choose an anisotropic form at the archimedean place.

In section 9, we combine all results established so far to compute the global period of the standard form with signature $(n + 2, 1)$ in $k = \mathbb{Q}$.

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Dedication. To the founder of IEEE–IST Academic (who went on to create IEEE Academic) and his accomplice in the darkest hour.

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1. SETUP (ISOTROPIC PLACES)

Let us recapitulate, from the previous paper [2], what happens at isotropic places.

Recall we fixed a number field k with adele ring \mathbb{A} , and a quadratic form $\langle \ , \ \rangle$ with matrix

$$\begin{pmatrix} 1 & & \\ & * & \\ & & -1 \end{pmatrix}$$

with respect to the decomposition $k^{n+3} = (k \cdot e_+) \oplus k^{n+1} \oplus (k \cdot e_-)$. We set $e = e_+ + e_-$ and named the following groups of isometries:

$$\begin{aligned} G &= O(n + 3), & \text{the isometry group of } \begin{pmatrix} * & * & * \end{pmatrix}; \\ H &= O(n + 2), & \text{the isometry group of } \begin{pmatrix} * & * & \end{pmatrix}; \\ \Theta &= O(n + 1), & \text{the isometry group of } \begin{pmatrix} & * & \end{pmatrix}. \end{aligned}$$

Let $P \subset G$ be the k -parabolic stabilizing $k \cdot e$. The modular function of P is given by $\delta_P(p) = |t|^{n+1}$ when $e \cdot p = e/t$. In particular, $\delta_P^s(p) = |t|^\alpha$, with $\alpha = (n + 1)s$.

We now choose an isotropic place v which, from this point onward, we will omit whenever possible. Therefore, k is the local field, \mathfrak{o} is its ring of integers, ϖ is a local uniformizer, and $\varpi = q^{-1}$ (q is the cardinality of the residue field).

Measure on $\Theta \backslash H$. Choose a hyperbolic pair x, x' in k^{n+2} so that $e_+ \in (k \cdot x) \oplus (k \cdot x')$ and change coordinates so that the restricted quadratic form has matrix

$$\begin{pmatrix} & & 1 \\ & B & \\ 1 & & \end{pmatrix}$$

with respect to the orthogonal decomposition $k^{n+2} = (k \cdot x') \oplus k^n \oplus (k \cdot x)$.

Let $Q \subset H$ be the parabolic stabilizing the line $k \cdot x$; we have

$$\int_{\Theta \backslash H} \text{function}(h) dh = \int_{\Theta \backslash \Theta Q} \text{function}(q) dq = \int_{(\Theta \cap Q) \backslash Q} \text{function}(q) dq.$$

Here, $\Theta \cap Q = O(n)$ is the fixer of x and x' . Set

$$m_\lambda = \begin{pmatrix} \lambda & & \\ & \text{id} & \\ & & 1/\lambda \end{pmatrix} \quad \text{and} \quad n_a = \begin{pmatrix} 1 & a & -\frac{1}{2}B(a) \\ & \text{id} & * \\ & & 1 \end{pmatrix}.$$

With $M_* = \{m_\lambda\}$, we have $\Theta \cap Q = \left\{ \begin{pmatrix} 1 & \\ & * \\ & 1 \end{pmatrix} \right\}$, $N^Q = \{n_a\}$, $M^Q = (\Theta \cap Q) \cdot M_*$, and $Q = M^Q \cdot N^Q$. The elements of $(\Theta \cap Q) \backslash Q$ can be expressed as $n_a \cdot m_\lambda$ and $\delta_Q(m_\lambda) = |\lambda|^n$. Moreover,

$$d(n_a \cdot m_\lambda) = da d\lambda$$

(with $d\lambda$ multiplicative and da additive) is a *right*-invariant measure. Therefore, up to a multiplicative constant independent of the integrand,

$$(2) \quad \int_{\Theta \backslash H} \text{function}(h) dh = \int_{k^\times} \int_{k^n} \text{function}(n_a \cdot m_\lambda) da d\lambda.$$

Construction of φ_ω . We saw in (1) that the local factor is

$$\int_{\Theta \backslash H} \varphi_\omega \cdot f,$$

where f generates an unramified principal series; in fact, $f(m_\lambda) = |\lambda|^\beta$, for some β (again, we are omitting the place v).

We restrict ourselves to the non-archimedean places.

We choose coordinates preserving the decomposition $k^{n+2} = (k \cdot x') \oplus k^n \oplus (k \cdot x)$ from above, and let $K^H \subset H$ be the compact open subgroup stabilizing integral (with respect to those coordinates) vectors. The details of what coordinates are actually chosen will transpire along the computation.

We want φ_ω to be associated with δ_P^s ; that is, if $e \cdot p = e/t$ and $\alpha = (n+1)s$, then $\varphi_\omega(p) = \omega(t) = |t|^\alpha$. Therefore, with Φ being the characteristic function of \mathfrak{o}^{n+3} , we define

$$\psi(g) = \int_{k^\times} \omega(t) \cdot \Phi(te \cdot g) dt \quad \text{and} \quad \varphi_\omega = \frac{\psi}{\psi(1)}.$$

The measure in k^\times is invariant with respect to multiplication normalized so that \mathfrak{o}^\times has volume 1.

Some notation. Let $|\varpi| = q^{-1}$ and $|t| = q^{-T}$. We will also use $a = q^{-\alpha}$, $z = q^{-\beta}$, and $w = zq^{-1} = q^{-\beta-1}$ and write (with the same measure as just above)

$$Z(\alpha) = \int_{k^\times \cap \mathfrak{o}} |t|^\alpha dt = \frac{1}{1 - q^{-\alpha}} = \frac{1}{1 - a}.$$

The integral converges only when $\operatorname{Re} \alpha$ is sufficiently large, but we will use $Z(\alpha)$ to denote the meromorphic continuation.

With $z = q^{-\beta}$, we define

$$\begin{aligned} X_\ell^B(\rho) &= \operatorname{meas} \left\{ a \in \mathfrak{o}^n : \frac{B(a) - \rho}{2} = 0 \pmod{\varpi^\ell} \right\}; \\ X^B(\beta; \rho) &= \sum_{\ell \geq 0} z^\ell X_\ell^B(\rho); \quad \text{and} \\ \Pi^B(\alpha, \beta) &= \int_{k^\times \cap \mathfrak{o}} |t|^\alpha X^B(\beta; t^2) dt. \end{aligned}$$

(When there is no risk of ambiguity, we suppress B or ρ , or use n instead of B .)

In order to make clear what adaptations are needed at even primes, we must repeat the following two proofs, with minor adjustments.

Proposition. *Up to a multiplicative constant independent of the integrand,*

$$\int_{\Theta \setminus H} \psi \cdot f = \int_{\Theta \setminus H} \int_{k^\times} |t|^\alpha \cdot \Phi(te \cdot h) \cdot f(h) dt dh = \Pi^n(\alpha - \beta - n, \beta).$$

(This is valid at all non-archimedean primes.)

Proof. According to (2), we have

$$\int_{\Theta \setminus H} \int_{k^\times} |t|^\alpha \Phi(te \cdot h) f(h) dt dh = \int_{k^\times} \int_{k^\times} \int_{k^n} |t|^\alpha |\lambda|^\beta \Phi(te \cdot n_a \cdot m_\lambda) da d\lambda dt.$$

At this point, we specify $e_+ = x' + \frac{1}{2}x$. Noting that

$$e \cdot n_a \cdot m_\lambda = (e_+ + e_-) \cdot n_a \cdot m_\lambda = e_+ \cdot n_a \cdot m_\lambda + e_-,$$

we have (in $k^{n+2} = (k \cdot x') \oplus k^n \oplus (k \cdot x)$)

$$e_+ \cdot n_a \cdot m_\lambda = \begin{pmatrix} 1 & 0 & \frac{1}{2} \end{pmatrix} \cdot n_a \cdot m_\lambda = \begin{pmatrix} \lambda & a & \frac{1}{2\lambda}(1 - B(a)) \end{pmatrix}$$

and (in $k^{n+3} = k^{n+2} \oplus (k \cdot e_-)$)

$$te \cdot n_a \cdot m_\lambda = \left(\lambda t, at, \frac{1}{2\lambda t}(t^2 - B(at)), t \right).$$

Therefore, after a change of variables,

$$\begin{aligned} \int_{\Theta \setminus H} \psi \cdot f &= \int_{k^\times} \int_{k^\times} \int_{k^n} |t|^{\alpha-\beta-n} |\lambda|^\beta \Phi\left(\lambda, a, \frac{1}{2\lambda}(t^2 - B(a)), t\right) da d\lambda dt \\ &= \int_{k^\times \cap \mathfrak{o}} |t|^{\alpha-\beta-n} \int_{k^\times \cap \mathfrak{o}} |\lambda|^\beta \int_{\mathfrak{o}^n} \operatorname{ch}_\mathfrak{o}\left(\frac{t^2 - B(a)}{2\lambda}\right) da d\lambda dt \\ &= \int_{k^\times \cap \mathfrak{o}} |t|^{\alpha-\beta-n} \sum_{\ell \geq 0} z^\ell X_\ell^n(t^2) = \int_{k^\times \cap \mathfrak{o}} |t|^{\alpha-\beta-n} X^n(\beta; t^2). \quad \square \end{aligned}$$

(3) **Proposition.** Fix $\varphi_\omega(p) = \delta_P(p)^s = \omega(t) = |t|^\alpha$ when $e \cdot p = e/t$ and $\alpha = (n+1)s$. Fix also a cuspidal F generating an irreducible $\pi = \otimes_v \pi_v$. Let f_v be a generator of π_v normalized by $f_v(1) = 1$, and let β_v be the local parameter of the unramified principal series representation π_v .

The local factor at the non-archimedean place v (omitted in the remainder of the statement) in the global period $(E_\varphi, F)_H$ is

$$\int_{\Theta \setminus H} \varphi_\omega \cdot f = \frac{1}{\psi(1)} \int_{\Theta \setminus H} \psi \cdot f = \frac{\Pi^n(\alpha - \beta - n, \beta)}{|2|^\alpha Z(\alpha)}$$

up to a multiplicative constant.

For the odd prime case, the multiplicative constant was determined in the previous paper, but the method does not seem applicable to even primes. In the cases for which we have computed Π , it depends only on the dimension of the anisotropic component and Witt index, the discriminant (whether it is a unit or is quadratic defect), and (for even primes) the Hasse–Minkowski invariant.

Proof. The multiplicative constant accounts for the normalization implied in the integral (2). Additionally,

$$\psi(1) = \int_{k^\times} |t|^\alpha \Phi(te) dt = \int_{k^\times} |t|^\alpha \Phi(t, 0, \frac{t}{2}, t) dt = \int_{k^\times \cap 2\mathfrak{o}} |t|^\alpha = |2|^\alpha Z(\alpha). \quad \square$$

Dimension reduction. By taking hyperbolic planes away, we can simplify the evaluation of (3) significantly. In fact, if there is a hyperbolic subspace with dimension $2k$ and $n = m + 2k$, then

$$\begin{aligned} X^{m+2k}(\beta; \rho) &= \frac{Z(\beta+1)}{Z(\beta+k+1)} \cdot X^m(\beta+k; \rho), \\ \Pi^{m+2k}(\alpha, \beta) &= \frac{Z(\beta+1)}{Z(\beta+k+1)} \cdot \Pi^m(\alpha, \beta+k). \end{aligned}$$

This is valid at all non-archimedean places.

All that is left to do, is to find the functions X and Π for anisotropic forms. The odd prime case was discussed in the previous paper. For even primes, we have anisotropic forms in k^m with $m \leq 4$.

2. EVEN PRIMES

The actual computation of X and Π (for anisotropic forms) at odd primes relied substantially on an anisotropy lemma, which guaranteed that certain equations had no solution modulo ϖ^ℓ . For even primes, we rely on a similar lemma.

In all that follows, $e = \text{ord } 2$ is the ramification index and $B(x) = \sum_i a_i x_i^2$ is a form with $0 \leq \text{ord } a_i \leq 1$.

(4) **Lemma.** Let $B(x) = \sum_i a_i x_i^2$ be anisotropic. Then, for each $x \neq 0$,

$$\max_i |a_i x_i^2| \geq |B(x)| \geq \max_i |4a_i x_i^2|.$$

Were that not the case, then the following lemma, with $\rho = B(x)$, would yield a nonzero solution of $B(x) = 0$.

(5) **Lemma.** *Let $B(x) = \sum_i a_i x_i^2$ be a form satisfying $0 \leq \text{ord } a_i \leq 1$. If there is a nonzero $x \in \mathfrak{o}^n$ and $\ell > \text{ord}(4a_i x_i^2)$ (for some i) such that*

$$B(x) = \rho \pmod{\varpi^\ell},$$

and if $|a_j x_j| = \max_i |a_i x_i|$, then there is $y \in x + (2a_j x_j)^{-1} \varpi^{\ell+1} \mathfrak{o}^n$ such that $B(y) = \rho$. In fact, $X_{\ell+1}^B(\rho) = |\varpi| X_\ell^B(\rho)$ if $\ell > \text{ord}(2\rho)$.

Proof. This follows from some versions of Hensel's lemma. We prove only the exact details we need in the continuation, as we will rely on specifics of the dyadic case.

Choose the highest $H \geq 0$ such that $x \in \varpi^H \mathfrak{o}^n$ and write $x = \varpi^H x'$, $y = \varpi^H y'$, $\rho = \varpi^{2H} \rho'$, and $\ell = 2H + \ell'$. Replacing x , y , ρ , and ℓ by x' , y' , ρ' , and ℓ' , both in the statement and in the conclusions, we see that we need address only the case $H = 0$, that is, the case $|x_j| = 1$.

Write $r = (2a_j)^{-1} \varpi^{\ell+1}$ and $y = x + ru$, leaving $u \in \mathfrak{o}^n$ unspecified. We have

$$\sum_i a_i y_i^2 = \sum_i a_i (x_i + ru_i)^2 = \sum_i a_i x_i^2 + \sum_i 2ra_i x_i u_i + \sum_i r^2 a_i u_i^2.$$

Because $|a_j| = |a_j x_j^2| = \max_i |a_i x_i^2|$, we have $|4a_j| = |4a_j x_j^2| \geq |4a_i x_i^2| > |\varpi^\ell|$, so

$$|r^2 a_i u_i^2| = |(2a_j)^{-2} \varpi^{2\ell+2} a_i u_i^2| = \left| \frac{\varpi^\ell}{4a_j} \frac{\varpi a_i}{a_j} \varpi^{\ell+1} u_i^2 \right| < |\varpi^{\ell+1}|,$$

Therefore, none of the $r^2 a_i y_i^2$ summands contributes modulo $\varpi^{\ell+1}$. On the other hand, from $|a_i x_i| \leq |a_j|$ we obtain

$$|2ra_i x_i u_i| = |a_j^{-1} \varpi^{\ell+1} a_i x_i u_i| = \left| \frac{a_i x_i}{a_j} \varpi^{\ell+1} u_i \right| \leq |\varpi^{\ell+1}|,$$

with equality (at least) when $i = j$ and $|u_j| = 1$.

That means that, no matter the choice for the other u_i , we can use u_j to control the value of $B(y)$ modulo $\varpi^{\ell+1}$. In other words, for exactly $|\varpi|$ (that is, one q th) of the choices of u_j (corresponding to $|\varpi|$ the possible refinements of x), we obtain

$$B(y) = \rho \pmod{\varpi^{\ell+1}},$$

a refinement of our original equation. Taking a limit, we obtain the desired solution.

If we know $\ell > \text{ord}(2\rho)$, we need no specifics on the values of x_i , so we can conclude $X_{\ell+1}^B(\rho) = |\varpi| X_\ell^B(\rho)$. The apparent mismatch between this statement and what was done above is due to the definition of X_ℓ involving an equation modulo $2\varpi^\ell$. \square

With $T = \text{ord } t \geq 0$, suppose now that $\rho = 4t^2 \neq 0 \pmod{2\varpi^\ell}$ (so, $\ell - 2T > e$), and that x is a solution of $B(x) = \rho \pmod{2\varpi^\ell}$. Then, according to lemma (4), $|4t^2| \geq \max_i |4a_i x_i^2|$; hence, $|t| \geq |x_i|$ and $x \in t\mathfrak{o}^n$. Therefore,

$$\begin{aligned} X_\ell(4t^2) &= \text{meas} \left\{ tx \in t\mathfrak{o}^n : B(tx) = 4t^2 \pmod{2\varpi^\ell} \right\} \\ &= |t|^n \text{meas} \left\{ x \in \mathfrak{o}^n : B(x) = 4 \pmod{2\varpi^{\ell-2T}} \right\} = |\varpi^n|^T X_{\ell-2T}(4). \end{aligned}$$

Still with $\ell > 2T + \text{ord } 2$, we also have, by the same reasoning,

$$X_\ell(0) = \text{meas} \left\{ tx \in t\mathfrak{o}^n : B(tx) = 4t^2 \pmod{2\varpi^\ell} \right\} = |\varpi^n|^T X_{\ell-2T}(0).$$

Finally, observing that $4t^2 = 0 \pmod{2\varpi^\ell}$ for $\ell \leq 2T + e$ and using $z = q^{-\beta}$ and $u = z^2 q^{-n}$, we obtain

$$\begin{aligned} X(\beta; 4t^2) - X(\beta; 0) &= \sum_{\ell > 2T+e} z^\ell X_\ell(4t^2) - \sum_{\ell > 2T+e} z^\ell X_\ell(0) \\ &= z^{2T} |\varpi^n|^T \sum_{k > e} z^k (X_k(4) - X_k(0)) \\ &= u^T \sum_{k \geq 0} z^k (X_k(4) - X_k(0)) = u^T (X(\beta; 4) - X(\beta; 0)). \end{aligned}$$

We have thus proven

$$X(\beta; 4\varpi^{2T}) = X(\beta; 0) + u^T X(\beta; 4) - u^T X(\beta; 0),$$

which leads to this conclusion:

(6) **Proposition.** *If $u = z^{-2\beta-n}$ and $a = q^{-\alpha}$, then*

$$\Pi(\alpha, \beta) = \sum_{0 \leq T < e} a^T X(\beta; \varpi^{2T}) + \frac{|2|^\alpha}{1-au} X(\beta; 4) + \frac{|2|^\alpha (a-au)}{(1-a)(1-au)} X(\beta; 0).$$

Proof. We have

$$\Pi(\alpha, \beta) = \sum_{0 \leq T < e} a^T X(\beta; \varpi^{2T}) + \sum_{T \geq 0} a^{T+e} X(\beta; 4\varpi^{2T}).$$

But

$$\begin{aligned} \sum_{T \geq 0} a^T X(\beta; 4\varpi^{2T}) &= \sum_{T \geq 0} a^T X(\beta; 0) + \sum_{T \geq 0} (au)^T X(\beta; 4) - \sum_{T \geq 0} (au)^T X(\beta; 0) \\ &= \frac{1}{1-au} X(\beta; 4) + \frac{(a-au)}{(1-a)(1-au)} X(\beta; 0). \quad \square \end{aligned}$$

Combining this proposition with lemma (5), we see that only finitely many values $X_\ell(t^2)$ need be computed.

Indeed, choose t with $1 \geq |t| = q^{-T} \geq |2|$ (this argument works for any $T \geq 0$). If $\ell > \text{ord}(2t^2) = 2T + e$ (so, at least for $\ell > \text{ord } 8$) and $k \geq 0$, the lemma tells us that $X_{\ell+k}(t^2) = |\varpi|^k X_\ell(t^2)$. Therefore,

$$X(\beta; t^2) = \sum_{0 \leq \ell \leq 2T+e} z^\ell X_\ell(t^2) + \sum_{k \geq 0} z^{2T+e+k+1} |\varpi|^k X_{2T+e+1}(t^2),$$

which, with $w = zq^{-1}$, simplifies to

$$X(\beta; t^2) = \sum_{0 \leq \ell < 2T+e+1} z^\ell X_\ell(t^2) + \frac{z^{2T+e+1}}{1-w} X_{2T+e+1}(t^2).$$

The anisotropy lemma (4) yields a similar reduction for $X_\ell(0)$: if $x \in \mathfrak{o}^n$ and $B(x) = 0 \pmod{2\varpi^\ell}$, then it must be that $|2\varpi^\ell| \geq \max_i |4a_i x_i^2|$, or $|2a_i x_i^2| \leq |\varpi^\ell|$. If $|\varpi^\ell| = |2\varpi^{2k+1}|$ or $|\varpi^\ell| = |2\varpi^{2k}|$, we may rely on $|x_i| \leq |\varpi^k|$, and in either case

$$X_\ell(0) = \text{meas} \left\{ \varpi^k x \in \varpi^k \mathfrak{o}^n : B(\varpi^k x) = 0 \pmod{2\varpi^\ell} \right\} = |\varpi^n|^k X_{\ell-2k}(0),$$

leading us to

$$\begin{aligned} \sum_{\ell \geq e} z^\ell X_\ell(0) &= \sum_{k \geq 0} z^{e+2k} X_{e+2k}(0) + \sum_{k \geq 0} z^{e+2k+1} X_{e+2k+1}(0) \\ &= z^e \sum_{k \geq 0} z^{2k} |\varpi^n|^k X_e(0) + z^{e+1} \sum_{k \geq 0} z^{2k} |\varpi^n|^k X_{e+1}(0). \end{aligned}$$

Using again $u = z^2 q^{-n} = q^{-2\beta-n}$, we obtain

$$X(\beta; 0) = \sum_{0 \leq \ell < e} z^\ell X_\ell(0) + \frac{z^e}{1-u} X_e(0) + \frac{z^{e+1}}{1-u} X_{e+1}(0).$$

In summary:

(7) **Proposition.** *If $z = q^{-\beta}$, $w = zq^{-1}$, $u = z^2 q^{-n}$, and $T \geq 0$, then*

$$\begin{aligned} X(\beta; \varpi^{2T}) &= \sum_{0 \leq \ell < 2T+e+1} z^\ell X_\ell(\varpi^{2T}) + \frac{z^{2T+e+1}}{1-w} X_{2T+e+1}(\varpi^{2T}); \\ X(\beta; 0) &= \sum_{0 \leq \ell < e} z^\ell X_\ell(0) + \frac{z^e}{1-u} X_e(0) + \frac{z^{e+1}}{1-u} X_{e+1}(0). \end{aligned}$$

We note that many of these values are repeated. For example, if $\ell \leq e$, then $X_\ell(0) = X_\ell(4)$. Additionally, if $\min_i |a_i| = 1$ (that is, all coefficients of the diagonal quadratic form are units), then the anisotropy lemma (4) implies $X_{e+1}(0) = |\varpi^n| X_{e-1}(0)$.

In practice, what we shall do is determine $X(\beta; \varpi^{2T})$ for all T when it takes no more effort than to do so only for $T \leq e$, or resort to proposition (7) otherwise.

3. CONICS IN DYADIC FIELDS

The computation of each $X_\ell(t^2)$ amounts to counting points modulo $2\varpi^\ell$ in conics. We discuss some preliminaries first.

We rely substantially on O'Meara's [23, §63] discussion of the quadratic defect in dyadic fields. We recall some of the relevant facts. The quadratic defect of ρ is the intersection of all ideals $b\mathfrak{o}$ for b such that $\rho - b$ is a square. If $b\mathfrak{o}$ is the quadratic defect of ρ , then $\eta^2 b\mathfrak{o}$ is the quadratic defect of $\eta^2 \rho$. If $\text{ord } \rho$ is odd, then the quadratic defect of ρ is $\rho\mathfrak{o}$. If $\text{ord } \rho$ is even, then the quadratic defect of ρ is 0 (if ρ is a square) or $4\rho\mathfrak{o}$, or $\rho\varpi^{2k+1}\mathfrak{o}$ with $0 \leq k < e$. If $\rho = \eta^2 + b$ is a unit and $0 < \text{ord } b = 2k + 1 < 2e$ or $0 < \text{ord } b = 2k < 2e$, then the quadratic defect of ρ is $\varpi^{2k+1}\mathfrak{o}$. The quotient of two units with quadratic defect $4\mathfrak{o}$ is a square. (Hence, half the units of the form $\eta^2 + b$ with $\text{ord } b = 2e$ are squares, and the other half have quadratic defect $4\mathfrak{o}$.) If $\rho = \eta^2 + b$ is a unit and $\text{ord } b > 2e$, then ρ is a square.

We recall that, for fixed dimension m , a form is classified by its discriminant Δ (we include the sign $(-1)^{\lfloor m/2 \rfloor}$ in its definition) and its Hasse–Minkowski invariant, built from the Hilbert symbol $(\ , \)$.

(8) **Lemma.** *Fix a non-square unit Δ .*

If the quadratic defect of Δ is $4\mathfrak{o}$, then $(a, \Delta) = (-1)^{\text{ord } a}$.

Otherwise, there is a unit a with quadratic defect $\varpi\mathfrak{o}$ such that $(a, \Delta) = -1$.

Proof. The first claim is proved by O'Meara [23, §63].

For the second claim, let $\varpi^d \mathfrak{o}$ be the quadratic defect of $\Delta = 1 + \varpi^d v$, as any other unit Δ with the same quadratic defect may be obtained with a change of variable in y . Write $a = 1 + \varpi u$, with u a unit. We want to show

$$ax^2 + \Delta y^2 = x^2 + \varpi ux^2 + y^2 + \varpi^d vy^2 = (x + y)^2 - 2xy + \varpi ux^2 + \varpi^d vy^2$$

is never a square, unless $x = y = 0$. If $|x| \neq |y|$, then the quadratic defect of the sum is the largest of $\varpi^d y^2 \mathfrak{o}$ and $\varpi x^2 \mathfrak{o}$. Therefore, we need only check the case $|x| = |y|$.

In the unramified case, use $\varpi = 2$ and $d = 1$. Let also $y = xt$. Then we want to choose u so that

$$ax^2 + \Delta y^2 = (x + xt)^2 + 2x^2(u - t + vt^2)$$

is never a square, or, which is the same, so that $u - t + vt^2 \not\equiv 0 \pmod{2}$ (were there any solutions of the latter equation, then we could refine at least one of the two so as to obtain a solution of the former).

But $t - vt^2$ is a separable quadratic polynomial, so in a finite field it takes only half the possible values, and we may choose for u any of the values it does not take. The same reasoning shows that $ux^2 - xy + vy^2 = 0 \pmod{2}$ has only one solution ($x = y = 0 \pmod{2}$) if and only if $(1 + 2u, 1 + 2v) = -1$.

In the ramified case with $d > 1$, we see $\text{ord}(-2xy + \varpi ux^2 + \varpi^d vy^2) = \text{ord}(\varpi ux^2)$ is odd, so $ax^2 + \Delta y^2$ is, indeed, never a square.

In the ramified case with $d = 1$, we may choose $a = \Delta$. Indeed, the reasoning above shows that $(\Delta, -1) = -1$ whenever $d < e$, so $(\Delta, \Delta) = (\Delta, -1) = -1$. \square

Here, we point out that if u is a unit, to say $1 + 4u$ is not a square is to say there is no unit v such that $(1 + 2v)^2 = 1 + 4v + 4v^2 = 1 + 4u$, which is to say $v + v^2 - u = 0$ is impossible, or $(1 + 2u, -1) = -1$ in the unramified case.

The first method. In its crudest form, the question we wish to answer is how many solutions there are to $x^2 = \rho \pmod{\varpi^\ell}$. Clearly, there are any if and only if ρ is a square modulo ϖ^ℓ . Most often, the number of solutions does not depend further on ρ ; in fact, if ρ is not a square, then the second case listed below does not occur.

(9) **Lemma.** *Let $X = \text{meas}\{x \in \mathfrak{o} : x^2 = \rho \pmod{\varpi^\ell}\}$, where $\rho \in \mathfrak{o}$ and $\ell \geq 0$. Write $\rho = \eta^2 + b$, where $b\mathfrak{o}$ is the quadratic defect of ρ .*

If $b \not\equiv 0 \pmod{\varpi^\ell}$, then $X = 0$.

If $b \equiv 0 \pmod{\varpi^\ell}$ and $|\varpi^\ell| < |4\eta^2|$, then $X = 2|\varpi^\ell/2\eta|$.

Otherwise, $X = |\varpi|^{\lceil \ell/2 \rceil}$.

Proof. The first claim is a consequence of the definition of quadratic defect.

The case $b \equiv 0 \pmod{\varpi^\ell}$ remains. We want to find solutions of $x^2 = \eta^2 \pmod{\varpi^\ell}$, which we rewrite as $(x - \eta)(x + \eta) = 0 \pmod{\varpi^\ell}$.

For the second claim, if $|\varpi^\ell| < |4\eta^2|$, then $b = 0$ and ρ is a square. The options $|x - \eta| = |x + \eta| = |2\eta|$ and $|x - \eta| > |2\eta|$ would lead to $|(x\eta)(x + \eta)| \geq |4\eta^2| > |\varpi^\ell|$. Hence, in order for x to be a solution, we require $|x \pm \eta| < |2\eta|$, in which case $|x \mp \eta| = |2\eta|$. That is, we are requiring $|x \pm \eta| \geq |\varpi^\ell/2\eta| < |2\eta|$. Therefore, $X = 2|\varpi^\ell/2\eta|$.

For the third claim, we consider $|\varpi^\ell| \geq |4\eta^2|$. If $|x - \eta| \leq |2\eta|$, then $|x + \eta| \leq |2\eta|$, so $|(x - \eta)(x + \eta)| \leq |4\eta^2| \leq |\varpi^\ell|$, so x is a solution. If $|x - \eta| > |2\eta|$, then

$|x + \eta| = |x - \eta|$, so $|(x - \eta)(x + \eta)| = |x - \eta|^2$, so x being a solution is equivalent to $|x - \eta|^2 \leq |\varpi^\ell|$. Therefore, $X = |\varpi|^{\lceil \ell/2 \rceil}$. \square

We will compute several sums of the form

$$\sum_{0 \leq \ell < L} z^\ell |\varpi|^{\lceil (\ell+o)/2 \rceil} = \sum_{0 \leq 2k < L} z^{2k} |\varpi|^{\lceil (2k+o)/2 \rceil} + \sum_{0 \leq 2k+1 < L} z^{2k+1} |\varpi|^{\lceil (2k+1+o)/2 \rceil}.$$

Set $w = z|\varpi|$. The first summand is

$$\sum_{0 \leq k < \lceil L/2 \rceil} (zw)^k |\varpi|^{\lceil o/2 \rceil} = |\varpi|^{\lceil o/2 \rceil} \frac{1 - (zw)^{\lceil L/2 \rceil}}{1 - zw},$$

while the second is

$$z|\varpi|^{\lceil (o+1)/2 \rceil} \frac{1 - (zw)^{\lfloor L/2 \rfloor}}{1 - zw} = w|\varpi|^{\lfloor o/2 \rfloor} \frac{1 - (zw)^{\lfloor L/2 \rfloor}}{1 - zw}.$$

Therefore,

$$(10) \quad \sum_{0 \leq \ell < L} z^\ell |\varpi|^{\lceil (\ell+o)/2 \rceil} = |\varpi|^{\lceil o/2 \rceil} \frac{1 - (zw)^{\lceil L/2 \rceil}}{1 - zw} + w|\varpi|^{\lfloor o/2 \rfloor} \frac{1 - (zw)^{\lfloor L/2 \rfloor}}{1 - zw}.$$

The second method. Though it is somewhat more versatile, this method can be used only in the unramified case. While discussing it, we always use $\varpi = 2$. We fix two units u and v such that $(1 + 2u, 1 + 2v) = -1$; as discussed in the proof of lemma (8), this is equivalent to saying that all solutions of $ux^2 + xy + vy^2 = 0 \pmod{2}$ satisfy $x = y = 0 \pmod{2}$.

Fix a quadratic polynomial $P(x, y) = ux^2 + Cxy + vy^2 + Bx + Ay + D \in \mathfrak{o}[x, y]$ with $C = 1 \pmod{2}$. Changing variables to $x = X + A$ and $y = Y + B$ and reducing modulo 2, we obtain

$$P(x, y) = uX^2 + XY + vY^2 + P(A, B) \pmod{2}.$$

(11) **Lemma.** *If $P(A, B) = 0 \pmod{2}$ and $\ell \geq 1$, then any solutions that may exist satisfy $x = A \pmod{2}$ and $y = B \pmod{2}$.*

If $P(A, B) \neq 0 \pmod{2}$ and $\ell \geq 1$, then

$$\text{meas}\{(x, y) \in \mathfrak{o}^2 : P(x, y) = 0 \pmod{2^\ell}\} = q^{-\ell} + q^{-\ell-1}.$$

Proof. We replace $x = X + A$ and $y = Y + B$. This has no effect on the measure.

When $\ell = 1$, the measure we want is

$$\text{meas}\{(X, Y) \in \mathfrak{o}^2 : uX^2 + XY + vY^2 = P(A, B) \pmod{2}\}.$$

If $P(A, B) = 0 \pmod{2}$, then $X = Y = 0 \pmod{2}$. Otherwise, $X = Y = 0 \pmod{2}$ is not a solution. Given a representative (X, Y) of a projective line (with respect to the residue field), there is exactly one (nonzero) value of $t \pmod{2}$ such that (Xt, Yt) is a solution. There are $q + 1$ projective lines, so the measure of the solution set is $(q + 1)/q^2 = q^{-1} + q^{-2}$.

For $\ell > 1$, suppose (X, Y) is a solution modulo 2^ℓ with Y a unit. Fix any refinement of Y modulo $2^{\ell+1}$. The coefficient of degree 1 in $P(X + A, Y + B) \in \mathfrak{o}[X]$ is a unit. Therefore, exactly one q th of the refinements of X modulo $2^{\ell+1}$ will yield a solution of the equation modulo $2^{\ell+1}$. The corresponding argument may be made if X is a unit. The effect in either case is

$$\text{meas}\{\text{solutions modulo } 2^{\ell+1}\} = |\varpi| \cdot \text{meas}\{\text{solutions modulo } 2^\ell\}. \quad \square$$

4. EVEN PRIMES— $m = 0$

If the original form is totally isotropic, we may reduce it to the case $m = 0$.

(12) **Proposition.** *Let $B = 0$ be the form in 0 variables. With $z = q^{-\beta}$ and $|t| = |\varpi|^T$, we have*

$$X(\beta; t^2) = \frac{1 - z^{2T+1-e}}{1 - z}$$

if $|t^2| \leq |2|$, and $X(\beta; t^2) = 0$ otherwise.

Proof. We want to evaluate

$$X_\ell(t^2) = \text{meas}\left\{0 : 0 = t^2 \pmod{2\varpi^\ell}\right\}.$$

The equation holds exactly if $t^2 = 0 \pmod{2\varpi^\ell}$, that is, if $2T \geq \ell + e$. \square

5. EVEN PRIMES— $m = 1$

(13) **Proposition.** *Let $B(x) = \Delta x^2$, where Δ is a nonsquare unit with quadratic defect $\varpi^d \mathfrak{o}$. With $z = q^{-\beta}$, $w = zq^{-1}$, and $|t| = q^{-T}$, we have*

$$X(\beta; t^2) = |\varpi|^{\lceil e/2 \rceil} \frac{1 - (zw)^{T + \lceil (d+1-e)/2 \rceil}}{1 - zw} + w|\varpi|^{\lfloor e/2 \rfloor} \frac{1 - (zw)^{T + \lfloor (d+1-e)/2 \rfloor}}{1 - zw}$$

if $|2| \geq |\varpi^d t^2|$, and $X(\beta; t^2) = 0$ otherwise.

Proof. According to lemma (9),

$$X_\ell(t^2) = \text{meas}\left\{x \in \mathfrak{o} : \Delta x^2 = t^2 \pmod{2\varpi^\ell}\right\}$$

(we use $2\varpi^\ell$ here, instead of ϖ^ℓ there) fails to be 0 only if $\varpi^d t^2 = 0 \pmod{2\varpi^\ell}$ (a unit Δ and its inverse have the same quadratic defect), that is, only when $0 \leq \ell < 2T + d - e + 1$. Using the final case of that lemma and applying (10), we obtain the answer. \square

(14) **Proposition.** *Let $B(x) = \Delta x^2$, where Δ is a unit square. With $z = q^{-\beta}$, $w = zq^{-1}$, and $|t| = q^{-T}$, we have*

$$X(\beta; t^2) = \frac{|\varpi|^{\lceil e/2 \rceil} + w|\varpi|^{\lfloor e/2 \rfloor}}{1 - zw} - \frac{(1+z)w^{e+1}}{1 - zw}(zw)^T + \frac{2(zw)^T w^{e+1}}{1 - w}.$$

Proof. According to lemma (9), $X_\ell(t^2)$ is different depending on whether $0 \leq \ell < 2T + e + 1$ or $\ell \geq 2T + e + 1$. In the first case, we obtain exactly the same sum as in the previous proof, but with $d = 2e$. Upon simplification, this yields the first two summands in the statement. For $\ell \geq 2T + e + 1$, lemma (9) tells us

$$\sum_{\ell \geq 2T+e+1} z^\ell X_\ell(t^2) = \sum_{\ell \geq 2T+e+1} z^\ell 2|\varpi^\ell/t| = \frac{2(zw)^T w^{e+1}}{1 - w}. \quad \square$$

(15) **Proposition.** *Let $B(x) = \Delta x^2$, where $|\Delta| = |\varpi|$. With $z = q^{-\beta}$, $w = zq^{-1}$, and $|t| = q^{-T}$, we have*

$$X(\beta; t^2) = |\varpi|^{\lfloor e/2 \rfloor} \frac{1 - (zw)^{T+1-\lfloor e/2 \rfloor}}{1 - zw} + z|\varpi|^{\lceil e/2 \rceil} \frac{1 - (zw)^{T-\lceil e/2 \rceil}}{1 - zw}$$

if $|2| \geq |t^2|$, and $X(\beta; t^2) = 0$ otherwise.

Proof. $X_\ell(t^2)$ fails to be 0 only if $t^2 = 0 \pmod{2\varpi^\ell}$, that is, only if $2T \geq \ell + e$, or $0 \leq \ell < 2T - e + 1$. In that case, lemma (9) tells us that $X_\ell(t^2) = |\varpi|^{\lceil(\ell+e-1)/2\rceil}$. The claimed outcomes follow from (10). \square

6. EVEN PRIMES— $m = 2$

We may write the anisotropic form as $B(x) = a(x_1^2 - \Delta x_2^2) = \sum_i a_i x_i^2$, where $|1| \geq |a|, |\Delta|, |a\Delta| \geq |\varpi|$ and $\Delta = \eta^2 + b$ has quadratic defect $b\mathfrak{o} = \varpi^d \mathfrak{o}$.

The Hasse–Minkowski invariant of such a form is $(a_1, a_2) = (a, -a\Delta) = (a, \Delta)$. We take $a = 1$ if we wish the invariant to be 1, or use lemma (8) if we wish it to be -1 .

Therefore, we have three situations for Δ : a unit with quadratic defect $4\mathfrak{o}$, or a unit with quadratic defect $\varpi^d \mathfrak{o}$ (d odd with $0 < d < 2e$), or else $|\Delta| = |\varpi|$. For each situation, we further distinguish the cases $(a, \Delta) = \pm 1$.

Before proceeding, we recall [23, §63] that, if a is a unit with quadratic defect $4\mathfrak{o}$, then $\{x^2 - ay^2 : x, y \in k\} = \{t \in k : \text{ord } t \text{ is even}\}$.

(16) **Proposition.** *Let $B(x) = x_1^2 - \Delta x_2^2$, where $|\Delta| = |\varpi|$. With $z = q^{-\beta}$, $w = zq^{-1}$, and $|t| = q^{-T}$, we have*

$$X(\beta; t^2) = |2| \frac{1 + w^{2T+e+1}}{1 - w}.$$

Proof. We have

$$X_\ell(t^2) = \text{meas}\left\{x \in \mathfrak{o}^2 : x_1^2 = \Delta x_2^2 + t^2 \pmod{2\varpi^\ell}\right\}.$$

According to lemma (9), in order to have a solution we need $\Delta x_2^2 = 0 \pmod{2\varpi^\ell}$ or (using the local square theorem and the fact that $\text{ord } \Delta$ is odd) $\Delta x_2^2 = 0 \pmod{4t^2}$.

If $4t^2 = 0 \pmod{2\varpi^\ell}$, we obtain

$$\begin{aligned} X_\ell &= \text{meas}\left\{x \in \mathfrak{o}^2 : x_1^2 = t^2 \pmod{2\varpi^\ell} \text{ and } \varpi x_2^2 = 0 \pmod{2\varpi^\ell}\right\} \\ &= |\varpi|^{\lceil(\ell+e)/2\rceil} \cdot |\varpi|^{\lceil(\ell+e-1)/2\rceil} = |\varpi|^{\ell+e}. \end{aligned}$$

If $4t^2 \neq 0 \pmod{2\varpi^\ell}$, we obtain

$$\begin{aligned} X_\ell &= \text{meas}\left\{x \in \mathfrak{o}^2 : x_1^2 = t^2 \pmod{2\varpi^\ell} \text{ and } \varpi x_2^2 = 0 \pmod{4t^2}\right\} \\ &= 2|\varpi|^{\ell+e}/2t \cdot |\varpi|^{e+T} = 2|\varpi|^{\ell+e}. \end{aligned}$$

Therefore,

$$X(\beta; t^2) = \sum_{\ell \geq 0} z^\ell |\varpi|^{\ell+e} + \sum_{\ell \geq 2T+e+1} z^\ell |\varpi|^{\ell+e} = |2| \frac{1 + w^{2T+e+1}}{1 - w}. \quad \square$$

(17) **Proposition.** *Let $B(x) = a(x_1^2 - \Delta x_2^2)$, where $|\Delta| = |\varpi|$ and a is a unit with quadratic defect $4\mathfrak{o}$. With $z = q^{-\beta}$, $w = zq^{-1}$, and $|t| = q^{-T}$, we have*

$$X(\beta; t^2) = |2| \frac{1 - w^{2T+e+1}}{1 - w}.$$

Proof. Because a is a unit, $x_1^2 - at^2$ yields exactly the elements of even degree, and the quadratic defect of a is $4\mathfrak{o}$, we have, consecutively,

$$\begin{aligned} X_\ell(t^2) &= \text{meas} \left\{ x \in \mathfrak{o}^2 : x_1^2 - at^2 = \Delta x_2^2 \pmod{2\varpi^\ell} \right\} \\ &= \text{meas} \left\{ x \in \mathfrak{o}^2 : \Delta x_2^2 = 4t^2 = 0 \pmod{2\varpi^\ell} \text{ and } x_1^2 = t^2 \pmod{2\varpi^\ell} \right\}. \end{aligned}$$

Therefore, $X_\ell(t^2)$ is nonzero only if $0 \leq \ell < 2T + e + 1$, in which case

$$X_\ell(t^2) = |\varpi|^{\lceil (\ell+e-1)/2 \rceil} \cdot |\varpi|^{\lceil (\ell+e)/2 \rceil} = |\varpi|^{\ell+e}. \quad \square$$

(18) **Proposition.** Let $B(x) = \varpi(x_1^2 - \Delta x_2^2)$, where Δ is a unit with quadratic defect $4\mathfrak{o}$. Let also $z = q^{-\beta}$, $w = zq^{-1}$, and $|t| = q^{-T}$.

If $2T < e$, then $X(\beta; t^2) = 0$.

If $2T \geq e$, write $(T - e)^+ = \max\{T - e, 0\}$ and $(T - e)^- = \min\{T - e, 0\}$. Then $X(\beta; t^2)$ is

$$\frac{|\varpi|^{\lfloor e/2 \rfloor} + z|\varpi|^{\lfloor e/2 \rfloor} - zw^e(zw)^{(T-e)^-}(w+1)}{1-zw} + \frac{w^e(z+w^2)(1-w^{2(T-e)^+})}{1-w^2}.$$

Proof. Because $x_1^2 - \Delta t^2$ yields exactly the elements of even degree and the quadratic defect of Δ is $4\mathfrak{o}$, we have, consecutively,

$$\begin{aligned} X_\ell(t^2) &= \text{meas} \left\{ x \in \mathfrak{o}^2 : \varpi(x_1^2 - \Delta x_2^2) = t^2 = 0 \pmod{2\varpi^\ell} \right\} \\ &= \text{meas} \left\{ x \in \mathfrak{o}^2 : t^2 = \varpi 4x_2^2 = 0 \pmod{2\varpi^\ell} \text{ and } \varpi x_1^2 = \varpi x_2^2 \pmod{2\varpi^\ell} \right\}. \end{aligned}$$

Considering only $0 \leq \ell < 2T - e + 1$, we obtain

$$X_\ell(t^2) = |\varpi|^{\lceil \max\{0, \ell-e-1\}/2 \rceil} \cdot |\varpi|^{\lceil (\ell+e-1)/2 \rceil}.$$

If $2T < e$, then $2T - e + 1 \leq 0$ always and $X(\beta; t^2) = 0$.

If $e \leq 2T < 2e + 1$, then $0 \leq \ell < 2T - e + 1 \leq e + 1$ always, and by (10),

$$\begin{aligned} X(\beta; t^2) &= \sum_{0 \leq \ell < 2T-e+1} z^\ell |\varpi|^{\lceil (\ell+e-1)/2 \rceil} \\ &= |\varpi|^{\lfloor e/2 \rfloor} \frac{1 - (zw)^{\lceil (2T-e+1)/2 \rceil}}{1-zw} + z|\varpi|^{\lfloor e/2 \rfloor} \frac{1 - (zw)^{\lfloor (2T-e+1)/2 \rfloor}}{1-zw}. \end{aligned}$$

If $2T \geq 2e + 2$, then

$$X(\beta; t^2) = \sum_{0 \leq \ell < e+1} z^\ell |\varpi|^{\lceil (\ell+e-1)/2 \rceil} + \sum_{e+1 \leq \ell < 2T-e+1} z^\ell |\varpi|^{2\lceil (\ell-e-1)/2 \rceil + e}.$$

The first sum is the same as before, but with T replaced by e . The second sum is obtained from (10) too (note we use $w^2 = z^2|\varpi|^2$ instead of $zw = z^2|\varpi|$):

$$zw^e \sum_{0 \leq \ell < 2T-2e} z^\ell |\varpi|^{2\lceil \ell/2 \rceil} = \frac{(zw^e + w^{e+2})(1 - w^{2T-2e})}{1 - w^2}. \quad \square$$

(19) **Proposition.** Let $B(x) = a(x_1^2 - \Delta x_2^2)$, where $\Delta = 1 + \varpi^d v$ is a unit with quadratic defect $\varpi^d \mathfrak{o}$, d is odd, $a = 1 + \varpi u$ is a unit with quadratic defect $\varpi \mathfrak{o}$, and $(a, \Delta) = -1$. Let also $z = q^{-\beta}$, $w = zq^{-1}$, and $|t| = q^{-T}$.

If $d = 1$ and $e > 1$, then

$$X(\beta; t^2) = |2| \frac{1 - w^{2T+2\lceil (e+1)/2 \rceil - e}}{1 - w}.$$

If $2T + 2 \geq e + 1 \geq d$ (with $d > 1$ or $e = 1$), then

$$X(\beta; t^2) = |2| |\varpi|^{(1-d)/2} \frac{1 - w^{2T+2-e}}{1 - w}.$$

If $2T + 2 \geq e + 1$ and $d > e + 1$, then

$$X(\beta; t^2) = \frac{|\varpi|^{\lceil e/2 \rceil} + w|\varpi|^{\lfloor e/2 \rfloor}}{1 - zw} - \frac{z^{d-e} |\varpi|^{(d+1)/2} (z - w)}{(1 - w)(1 - zw)} - \frac{w^{2T+2-e} |\varpi|^{e+(1-d)/2}}{1 - w}.$$

If $2T + 2 \leq e$ (with $d > 1$), then $X(\beta; t^2) = 0$.

Proof. We have

$$X_\ell(t^2) = \text{meas} \left\{ x \in \mathfrak{o}^2 : x_1^2 = \Delta x_2^2 + at^2 \pmod{2\varpi^\ell} \right\}.$$

If $|t| > |x_2|$, we need $\varpi t^2 = 0 \pmod{2\varpi^\ell}$.

If $|t| < |x_2|$, we need $\varpi t^2 = \varpi^d x_2^2 = 0 \pmod{2\varpi^\ell}$.

If $|t| = |x_2|$, we write $x_2 = tz$ (with z a unit) and observe

$$\Delta x_2^2 + at^2 = t^2((1 + z)^2 + \varpi u - 2z + \varpi^d v z^2).$$

Therefore, if $d > 1$ we require $\varpi t^2 = 0 \pmod{2\varpi^\ell}$. If $d = e = 1$ and $\varpi = 2$, and because $(1 + 2u, 1 + 2v) = -1$, we always have $u - z + v z^2 \not\equiv 0 \pmod{2}$, so we require $2t^2 = 0 \pmod{2\varpi^\ell}$. Finally, if $d = 1$ and $e > 1$, then we can choose $v = u$ and write

$$\Delta x_2^2 + at^2 = t^2(1 + \varpi u)(1 + z^2) = t^2((1 + \varpi u)(1 + z)^2 - 2(1 + \varpi u)z);$$

we thus require $t^2 \varpi(1 + z)^2 = 0 \pmod{2\varpi^\ell}$ and either $2\varpi t^2 = 0 \pmod{2\varpi^\ell}$ (if e is even) or $2t^2 = 0 \pmod{2\varpi^\ell}$ (if e is odd).

If $d = 1$ and $e > 1$, we required $\varpi t^2 = \varpi x_2^2 = 0 \pmod{2\varpi^\ell}$ (when $|t| \neq |x_2|$) or $\varpi(t + x_2)^2 = 0 \pmod{2\varpi^\ell}$ and $\varpi^{2\lceil (e+1)/2 \rceil - e - 1} t^2 = 0 \pmod{\varpi^\ell}$ (when $|t| = |x_2|$). We obtain

$$X_\ell(t^2) = |\varpi|^{\lceil (\ell+e)/2 \rceil} \cdot |\varpi|^{\lceil (\ell+e-1)/2 \rceil} = |\varpi|^{\ell+e}$$

if $\ell < 2T + 2\lceil (e+1)/2 \rceil - e$ and $X_\ell(t^2) = 0$ otherwise.

If $d > 1$ or $e = 1$, we required $\varpi t^2 = \varpi^d x_2^2 = 0 \pmod{2\varpi^\ell}$. Therefore,

$$X_\ell(t^2) = |\varpi|^{\lceil (\ell+e)/2 \rceil} \cdot |\varpi|^{\max\{0, \lceil (\ell+e-d)/2 \rceil\}}$$

if $\ell < 2T + 2 - e$ and $X_\ell(t^2) = 0$ otherwise. \square

So far, we have relied on the first method discussed in section 3. For all remaining quadratic forms, we will use the second one. In particular, all that follows is valid only for the unramified case.

The strategy is always the same: we first reduce the equation modulo 2. This corresponds to $\ell = 0$ and suggests a substitution for one of the variables. That variable will be set modulo 2—hence, we always have an extra factor q^{-1} in the final calculation of $X_\ell(t^2)$.

Applying the substitution and simplifying, we obtain a new equation, modulo 2^ℓ (the original equation was modulo $2^{\ell+1}$). At this point, we consider the case $\ell = 1$. If the equation thus reduced is linear with unit coefficient, we know how many solutions it has. If the equation is quadratic, we apply lemma (11): either we obtain new conditions on other variables, typically allowing us to divide the original equation by 4 and conclude $X_\ell(t^2) = q^{-m} X_{\ell-2}(t^2/4)$, or we obtain a solution count.

(20) **Proposition.** *Let $B(x) = x_1^2 - \Delta x_2^2$, where $\Delta = 1 + 4v$ is a unit with quadratic defect $4\mathfrak{o}$ and v is a unit. In the unramified case, with $z = q^{-\beta}$, $w = zq^{-1}$, and $|t| = q^{-T}$, we have*

$$X(\beta; 1) = \frac{|2|}{1-w} \quad \text{and} \quad X(\beta; 4t^2) = |2| \frac{1+z+w^{2T}(zw+w^3)}{1-w^2}.$$

Proof. The equation is $x_1^2 - \Delta x_2^2 = t^2 \pmod{2^{\ell+1}}$. Considering $\ell = 0$, we are led to $x_1 = x_2 + t + 2b$, for some $b \in \mathfrak{o}$. We substitute and simplify:

$$2b^2 + x_2 t + 2x_2 b + 2tb - 2vx_2^2 = 0 \pmod{2^\ell}.$$

If $t = 1$ and $\ell \geq 1$, we clearly can obtain a unique $x_2 \pmod{2^\ell}$. Therefore, recalling x_1 was set modulo 2, we have $X_\ell(1) = q^{-1-\ell}$.

If $2 \mid t$, the equation holds for $\ell \leq 1$; that is, $X_0(t^2) = X_1(t^2) = q^{-1}$. If $\ell \geq 2$, we divide further:

$$b^2 + x_2 \frac{t}{2} + x_2 b + tb - vx_2^2 = 0 \pmod{2^{\ell-1}}.$$

We note that $(1 + 2v, -1) = -1$, so we may apply lemma (11).

For $t = 2$, the lemma tells us the measure of the solution set with respect to b and x_2 is $q^{-\ell+1} + q^{-\ell}$, so with respect to x_1 and x_2 , for $\ell \geq 2$, we have

$$X_\ell(4) = q^{-\ell} + q^{-\ell-1}.$$

If $4 \mid t$ and $\ell \geq 2$, then $x_2 = b = x_1 = 0 \pmod{2}$, so $X_\ell(t^2) = q^{-2} X_{\ell-2}(t^2/4)$. \square

(21) **Proposition.** *Let $B(x) = x_1^2 - \Delta x_2^2$, where $\Delta = 1 + 2v$ is a unit with quadratic defect $2\mathfrak{o}$ and v is a unit. In the unramified case, with $z = q^{-\beta}$, $w = zq^{-1}$, and $|t| = q^{-T}$, we have*

$$X(\beta; t^2) = |2| \frac{1 + w^{2T+1}}{1-w}.$$

Proof. The equation is $x_1^2 - \Delta x_2^2 = t^2 \pmod{2^{\ell+1}}$. Replacing $x_1 = x_2 + t + 2b$ and simplifying:

$$2b^2 + x_2 t + 2x_2 b + 2tb - vx_2^2 = 0 \pmod{2^\ell}.$$

If $t = 1$, we have $x_2 = 0 \pmod{2}$ or $x_2 = v^{-1} \pmod{2}$, and, because the coefficient of x_2 is a unit, solutions can be refined modulo 2^ℓ . Therefore, $X_0(1) = q^{-1}$ and $X_\ell(1) = q^{-1} \cdot 2q^{-\ell} = 2q^{-\ell-1}$ for $\ell \geq 1$.

If $2 \mid t$, then $x_2 = x_1 = 0 \pmod{2}$. Therefore, $X_0(t^2) = q^{-1}$, $X_1(t^2) = q^{-2}$, and $X_\ell(t^2) = q^{-2} X_{\ell-2}(t^2/4)$ for $\ell \geq 2$. \square

7. EVEN PRIMES— $m = 3$

A ternary quadratic form with discriminant Δ is anisotropic if and only if its Hasse–Minkowski invariant is $(-1, \Delta)$.

The form $B(x) = x_1^2 - a(x_2^2 - \Delta x_3^2)$ has discriminant Δ and Hasse–Minkowski invariant $(-a, a\Delta) = (-a, \Delta) = (-1, \Delta)(a, \Delta)$. Therefore, it is anisotropic when $(a, \Delta) = -1$. If $|\Delta| = |\varpi|$, we may take any a with quadratic defect $4\mathfrak{o}$. If Δ is a unit with quadratic defect $\varpi^d \mathfrak{o}$ (with d odd), lemma (8) yields a unit a with quadratic defect $\varpi \mathfrak{o}$ and $(a, \Delta) = -1$.

If Δ is a unit square or a unit with quadratic defect $4\mathfrak{o}$, the form is anisotropic if and only if the Hasse–Minkowski invariant is -1 . We choose $a = 1 + \varpi u$, where u is a unit and $(a, -1) = -1$, provided by lemma (8). The form $B(x) = a(x_1^2 + x_2^2) - \Delta x_3^2$ has discriminant Δ and Hasse–Minkowski invariant $(a, a) = (a, -1) = -1$.

We are still using the second method, so we consider only the unramified case.

(22) **Proposition.** *Let $B(x) = x_1^2 - a(x_2^2 - 2vx_3^2)$, where $a = 1 + 4u$ is a unit with quadratic defect $4\mathfrak{o}$, and u and v are units. In the unramified case, with $z = q^{-\beta}$, $w = zq^{-1}$, and $|t| = q^{-T}$, we have*

$$X(\beta; 1) = \frac{|2|}{1 - w}$$

and

$$X(\beta; t^2) = |2| \frac{1 + w}{1 - w^2 q^{-1}} + w^{2T} q^{-T} \frac{1 - w^2 q^{-2}}{(1 - w)(1 - w^2 q^{-1})}, \quad \text{if } T \geq 1.$$

Proof. The equation is $x_1^2 - a(x_2^2 - 2vx_3^2) = t^2 \pmod{2^{\ell+1}}$. We substitute $x_1 = x_2 + t + 2b$, simplify, and obtain

$$2b^2 - x_2 t - 2x_2 b - 2tb - 2ux_2^2 + avx_3^2 = 0 \pmod{2^\ell}.$$

If $t = 1$ and x_3 is fixed, we obtain exactly one solution $x_2 \pmod{2^\ell}$. Therefore, $X_\ell(1) = q^{-1} \cdot q^{-\ell} = q^{-\ell-1}$.

If $2 \mid t$ and $\ell \geq 1$, we are led to $x_3 = 0 \pmod{2}$. For such t , we have $X_0(t^2) = q^{-1}$ and $X_1(t^2) = q^{-2}$. If $\ell \geq 2$, we have also

$$b^2 - x_2 \frac{t}{2} - x_2 b - 2\frac{t}{2}b - ux_2^2 + 2av\frac{x_3^2}{4} = 0 \pmod{2^{\ell-1}}.$$

If $t = 2$, we obtain $q^{-\ell+1} + q^{-\ell}$ solutions with respect to b and x_2 , or $q^{-\ell} + q^{-\ell-1}$ with respect to x_1 and x_2 . Recalling that $x_3 = 0 \pmod{2}$, we see that, for $\ell \geq 2$, $X_\ell(4) = q^{-\ell-1} + q^{-\ell-2}$.

If $4 \mid t$, then we obtain $x_2 = 0 \pmod{2}$, so $x_1 = x_2 = x_3 = t = 0 \pmod{2}$, and we have, for $\ell \geq 2$, $X_\ell(t^2) = q^{-3} X_{\ell-2}(t^2/4)$. \square

(23) **Proposition.** *Let $B(x) = x_1^2 - a(x_2^2 - \Delta x_3^2)$, where $a = 1 + 2u$ and Δ are units with quadratic defect $2\mathfrak{o}$, $(a, \Delta) = -1$, $-a\Delta = 1 + 2v$, and u and v are units. In the unramified case, with $z = q^{-\beta}$, $w = zq^{-1}$, and $|t| = q^{-T}$, we have*

$$X(\beta; t^2) = |2| \frac{1 + wq^{-1}}{1 - w^2 q^{-1}} + |2| w^{2T+1} q^{-T} \frac{1 - w^2 q^{-2}}{(1 - w)(1 - w^2 q^{-1})}.$$

Proof. The equation is $x_1^2 - ax_2^2 + a\Delta x_3^2 = t^2 \pmod{2^{\ell+1}}$. Replacing $x_1 = x_2 + x_3 + t + 2b$ and simplifying, we obtain

$$2b^2 + x_2 x_3 + x_2 t + 2x_2 b + x_3 t + 2x_3 b + 2tb - ux_2^2 - vx_3^2 = 0 \pmod{2^\ell}.$$

In particular, $X_0(t^2) = q^{-1}$.

If $t = 1$ and $\ell \geq 1$, and recalling we already set $x_1 \pmod{2}$, we obtain $X_\ell(1) = q^{-\ell-1} + q^{-\ell-2}$.

If $2 \mid t$ and $\ell \geq 1$, we conclude $x_2 = x_3 = 0 \pmod{2}$, so $X_1(t^2) = q^{-3}$ and, for $\ell \geq 2$, $X_\ell(t^2) = q^{-3} X_{\ell-2}(t^2/4)$. \square

(24) **Proposition.** *Let $B(x) = a(x_1^2 + x_2^2) - x_3^2$, where $a = 1 + 2u$ is a unit with quadratic defect $2\mathfrak{o}$, $(a, -1) = -1$, and u is a unit. In the unramified case, with $z = q^{-\beta}$, $w = zq^{-1}$, and $|t| = q^{-T}$, we have*

$$X(\beta; t^2) = |2| \frac{(1 + wq^{-1})(1 - w^{2T+2} q^{-T-1})}{1 - w^2 q^{-1}}.$$

Proof. The equation is $x_1^2 + x_2^2 - a\Delta x_3^2 = at^2 \pmod{2^{\ell+1}}$. Replacing $x_1 = x_2 + x_3 + t + 2b$ and simplifying, we obtain

$$x_2^2 + 2b^2 + x_2x_3 + x_2t + 2x_2b + x_3t + 2x_3b + 2tb - ux_3^2 - ut^2 = 0 \pmod{2^\ell}.$$

In particular, $X_0(t^2) = q^{-1}$.

If $t = 1$ and $\ell = 1$, lemma (11) tells us $x_2 = x_3 = 1 \pmod{2}$ and $X_1(1) = q^{-3}$.

If $t = 1$ and $\ell = 2$, taking into account that $x_2 = x_3 = 1 \pmod{2}$, we see the equation is equivalent to

$$b^2 - b - u = 0 \pmod{2}.$$

This equation has no solution, therefore, $X_\ell(1) = 0$ for $\ell \geq 2$.

If $2 \mid t$ and $\ell \geq 1$, then the equation leads to $x_2 = x_3 = t = x_1 = 0 \pmod{2}$. This means $X_1(t^2) = q^{-3}$ and $X_\ell(t^2) = q^{-3}X_{\ell-2}(t^2/4)$ if $\ell \geq 2$. \square

(25) **Proposition.** Let $B(x) = a(x_1^2 + x_2^2) - \Delta x_3^2$, where $\Delta = 1 + 4u$ is a unit with quadratic defect $4\mathfrak{o}$, $a = 1 + 2u$ is a unit with quadratic defect $2\mathfrak{o}$, u is a unit, and $(a, -1) = -1$. In the unramified case, with $z = q^{-\beta}$, $w = zq^{-1}$, and $|t| = q^{-T}$, we have

$$X(\beta; t^2) = |2| \frac{1 + wq^{-1}}{1 - w^2q^{-1}} + |2| w^{2T+2} q^{-T-1} \frac{(1+w)(1-wq^{-1})}{(1-w)(1-w^2q^{-1})}.$$

Proof. The equation is $x_1^2 + x_2^2 - a\Delta x_3^2 = at^2 \pmod{2^{\ell+1}}$. Replacing $x_1 = x_2 + x_3 + t + 2b$ and simplifying, we obtain

$$x_2^2 + 2b^2 + x_2x_3 + x_2t + 2x_2b + x_3t + 2x_3b + 2tb - u(3 + 4u)x_3^2 - ut^2 = 0 \pmod{2^\ell}.$$

Clearly, $X_0(t^2) = q^{-1}$.

If $t = 1$, the equation, reduced modulo 4, is equivalent to

$$(x_2 + 1)^2 + (x_2 + 1)(x_3 - 1) + 2b(b + x_2 + x_3 + 1) + u(x_3 - 1)^2 + 2u(x_3 - 1) = 0 \pmod{4}.$$

If $\ell \geq 1$, we must have $x_2 = x_3 = 1 \pmod{2}$, so $X_1(1) = q^{-3}$. If $\ell \geq 2$, we must also have $b^2 = b \pmod{2}$, that is, $b = 0 \pmod{2}$ or $b = 1 \pmod{2}$, which leads to $X_2(1) = 2q^{-4}$. Therefore, $X_\ell(1) = 2q^{-\ell-2}$ for $\ell \geq 2$.

If $2 \mid t$, the original equation yields $x_2 = x_3 = t = x_1 = 0 \pmod{2}$. Therefore, $X_1(t^2) = q^{-3}$ and $X_\ell(t^2) = q^{-3}X_{\ell-2}(t^2/4)$ if $\ell \geq 2$. \square

8. EVEN PRIMES— $m = 4$

In the $m = 4$ case, we have only one equivalence class of anisotropic forms.

(26) **Proposition.** Let $B(x) = x_1^2 + x_2^2 - a(x_3^2 + x_4^2)$, where $a = 1 + 2u$, u is a unit, and $(a, -1) = -1$. In the unramified case, with $z = q^{-\beta}$, $w = zq^{-1}$, and $|t| = q^{-T}$, we have

$$X(\beta; 1) = \frac{|2|}{1 - w}$$

and

$$X(\beta; t^2) = \frac{|2|}{1 - wq^{-1}} + w^{2T} q^{-2T} \frac{1 - wq^{-2}}{(1 - w)(1 - wq^{-1})}, \quad \text{if } T \geq 1.$$

Proof. We know $x_1 + x_2 + x_3 + x_4 + t = 0 \pmod{2}$. Replacing $x_1 = Z + t$ and $x_2 = x_3 + x_4 + Z + 2b$ into $B(x) = t^2 \pmod{2^{\ell+1}}$ and simplifying, we obtain

$$-ux_3^2 + x_3x_4 - ux_4^2 + (Z + 4b)x_3 + (Z + 4b)x_4 + Z^2 + Zt + 4bZ + 4b^2 = 0 \pmod{2^\ell}.$$

As usual, $X_0(t^2) = q^{-1}$.

If $t = 1$, we only need to reduce the equation modulo 4, we obtain

$$(27) \quad -ux_3^2 + x_3x_4 - ux_4^2 + Zx_3 + Zx_4 + Z^2 + Zt = 0 \pmod{4}.$$

Replacing $x_3 = x_4 = Z \pmod{2}$, we obtain $Zt \pmod{2}$.

If $t = 1$, $\ell = 1$, and Z is a unit, then the solution set (with respect to x_3 and x_4 , and modulo 2) has measure $q^{-1} + q^{-2}$. If Z is not a unit, then $x_3 = x_4 = 0 \pmod{2}$. Therefore (we must not forget that x_2 was set modulo 2),

$$X_1(1) = q^{-1}(1 - q^{-1})(q^{-1} + q^{-2}) + q^{-4} = q^{-2}.$$

If $t = 1$ and $\ell = 2$, and Z is a unit, then the solution set (with respect to x_3 and x_4 , and modulo 4) has measure $q^{-2} + q^{-3}$. If Z is not a unit, then $x_3 = x_4 = 0 \pmod{2}$ and $Z = 0 \pmod{4}$. Therefore (again, x_2 was set modulo 2),

$$X_2(1) = q^{-1}(1 - q^{-1})(q^{-2} + q^{-3}) + q^{-5} = q^{-3}.$$

More generally, $X_\ell(1) = q^{-\ell-1}$.

If $2 \mid t$, we return to equation (27). Lemma (11) tells us $x_3 = x_4 = Z \pmod{2}$, so $X_1(t^2) = q^{-3}$ (again, we must not forget x_2 was set modulo 2). For $\ell \geq 2$, we substitute $x_3 = Z + 2X$ and $x_4 = Z + 2Y$ and simplify:

$$-uZ^2 + Z\frac{t}{2} = 0 \pmod{2}.$$

If $t = 2$ and $Z = 0 \pmod{2}$, we conclude $x_1 = x_2 = x_3 = x_4 = t = 0 \pmod{2}$, so the contribution (in the equation modulo 2^ℓ) is $q^{-4}X_{\ell-2}(1) = q^{-3-\ell}$. If $t = 2$ and $Z = u^{-1} \pmod{2}$, then Z is determined modulo $2^{\ell-1}$ (recall we divided by 2 in most recent simplification), so the contribution is $q^{-\ell+1}q^{-1}q^{-1}q^{-1} = q^{-2-\ell}$. Therefore, $X_\ell(4) = q^{-2-\ell} + q^{-3-\ell}$.

If $4 \mid t$ (still for $\ell \geq 2$), we may also conclude $Z = 0 \pmod{2}$, so $X_\ell(t^2) = q^{-4}X_{\ell-2}(t^2/4)$. \square

9. SOME EXAMPLES IN $k = \mathbb{Q}$

In order to determine the global period in all cases, we still need the local factors at some even primes (the ramified cases we could not address here), the missing normalization constant in proposition (3), and the local factors at archimedean primes.

Therefore, for these examples we will ignore multiplicative constants and consider only $k = \mathbb{Q}$ and the standard form $\sum_{i=1}^{n+2} x_i^2 - x_{n+3}^2$ with signature $(n+2, 1)$ on $k^{n+2} \oplus k \cdot e_-$.

At the archimedean place, $k_v = \mathbb{R}$ and the restriction of the form to \mathbb{R}^{n+2} is anisotropic. Hence, as mentioned in the introduction, the local factor of the period is simply $\text{vol}(\Theta_v \setminus H_v) = \text{vol}(\text{O}(n+1, \mathbb{R}) \setminus \text{O}(n+2, \mathbb{R}))$, a multiplicative constant.

At non-archimedean places, we have simply $q_v = p$ (where p is the prime). As the discriminant is $\Delta = \pm 1$, there are no bad odd primes.

If $\Delta = 1$, the associated quadratic character is the trivial character χ_0 (its L -function is the Riemann zeta function). If $\Delta = -1$, the associated quadratic character is the character χ_1 given by $\chi_1(p) = 1$ if $p \equiv 1 \pmod{4}$, $\chi_1(p) = -1$ if $p \equiv 3 \pmod{4}$, or $\chi_1(p) = 0$ if p is even. Also,

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} \quad \text{and} \quad L(s, \chi) = \prod_p \frac{1}{1 - \chi(p)p^{-s}}.$$

We will also limit our attention to periods $(E_\varphi, 1)_H$, that is, periods of the Eisenstein series alone, rather than against a cuspidal F , in which case the local parameters are $\beta = 0$.

With $\alpha = (n+1)s$, we saw before [2] that, up to multiplicative constants and correction factors at $p = 2$ (determined in this paper), the global period is

$$(28) \quad (E_\varphi, 1)_H = \frac{\zeta(\alpha - n)}{L(\alpha - \lfloor \frac{n}{2} \rfloor, \chi)}, \quad \text{if } n \text{ is odd;}$$

$$(29) \quad (E_\varphi, 1)_H = \frac{\zeta(\alpha - n)}{\zeta(2\alpha - n)/L(\alpha - \frac{n}{2}, \chi)}, \quad \text{if } n \text{ is even}$$

(where $\chi = \chi_0$ when $\Delta = 1$, and $\chi = \chi_1$ when $\Delta = -1$).

Say B^{n+2} is the original form with signature $(n+2, 0)$ and B^n is the original form B (from the discussion, in section 1, of the measure on $\Theta \backslash H$). Say also that B^{n-2k} is the form obtained after taking k hyperbolic planes away (for the dimension reduction cited at the end of section 1), until we get an anisotropic form B^m , with $n = m + 2k$ —those are the forms whose X functions we computed in sections 4 through 8.

All those forms have the same discriminant Δ . However, their Hasse–Minkowski invariants are not the same. Let $\text{hmi } B$ denote the Hasse–Minkowski invariant of a form B . In general [4, 23], if B is the sum of two forms C and D , then $\text{hmi } B = (\det C, \det D) \cdot \text{hmi } C \cdot \text{hmi } D$.

A hyperbolic plane has determinant -1 and invariant $(1, -1) = 1$. Also, B^{n+2} has invariant 1 and determinant 1, so $\det B^{n-2k} = (-1)^{k+1}$ and $(-1, \det B^{n-2k}) = (-1)^{k+1}$. This means that with even k we change the sign of the invariant, and with odd k we keep it. That is, starting with $\text{hmi } B^{n+2}$, and taking one hyperbolic plane away at a time, we obtain 1, -1 , -1 , 1, and then repeat with period four.

Applying the discussions at the beginning of sections 6 through 8 to our current case, we see that B^2 is anisotropic if and only if $\Delta = -1$, that B^3 is anisotropic if and only if $\text{hmi } B^3 = -\Delta$, and that B^4 is anisotropic if and only if $\Delta = \text{hmi } B^4 = 1$.

We thus obtain the information in table 1, for $n \geq 3$. With $n < 3$, H would be anisotropic at $p = 2$, the local factor would be a constant, and the results in this paper would not be used. Taking our choice $\beta = 0$ and the dimension reduction at the end of section 1 into account, we use $z = q^{-k}$ and $w = q^{-k-1}$ and abbreviate $u = q^{-n}$ (this is always what is raised to the power T). Additionally, as the local factor is obtained by integration of this X with respect to t (that is, in terms of these formulas, a summation with respect to $T = \text{ord } t$) and we are missing a multiplicative constant, we multiply by a common factor so that the result is as close as possible to the form $1 - Au^T$.

That choice simplifies substantially the computation of Π . Indeed, with $a = q^{-\alpha}$, the definition of Π (from section 1, using the multiplicative measure) is

$$\Pi(\alpha, \beta) = \int_{k^\times \cap \mathfrak{o}} |t|^\alpha X(\beta; t^2) dt = \sum_{T \geq 0} X(\beta; t^2) a^T.$$

If, up to the common factors we dropped, $X = 1 - Au^T$, this becomes

$$\Pi(\alpha, \beta) = \sum_{T \geq 0} (a^T - A(au)^T) = \frac{1}{1-a} - \frac{A}{1-au} = \frac{(1-A) - a(u-A)}{(1-a)(1-au)}.$$

TABLE 1. Taking k hyperbolic planes from B^n , we obtain the anisotropic form B^m , whose X function with list, as well as the respective proposition.

n	Δ	k	m	hmi B^m	prop.	$X^m(\beta; t^2)$
$3 + 8\ell$	1	4ℓ	3	-1	(24)	$1 - uu^T$;
$4 + 8\ell$	-1	$4\ell + 1$	2	-1	(19)	$1 - wu^T$;
$5 + 8\ell$	-1	$4\ell + 2$	1	1	(13)	$1 - \frac{u+z}{1+z}u^T$;
$6 + 8\ell$	1	$4\ell + 3$	0	1	(12)	$\begin{cases} 1, & \text{if } T = 0, \\ 0, & \text{otherwise;} \end{cases}$
$7 + 8\ell$	1	$4\ell + 3$	1	1	(14)	$1 + \frac{1-w-u}{1+w-u}uu^T$;
$8 + 8\ell$	-1	$4\ell + 3$	2	1	(21)	$1 + wu^T$;
$9 + 8\ell$	-1	$4\ell + 3$	3	1	(23)	$1 - \frac{u-w}{1-w}u^T$;
$10 + 8\ell$	1	$4\ell + 3$	4	1	(26)	$\begin{cases} \frac{1-wq^{-1}}{1-w}, & \text{if } T = 0, \\ 1 + \frac{1-wq^{-2}}{q^{-1}(1-w)}u^T, & \text{otherwise.} \end{cases}$

This becomes even simpler when $A = \frac{u-v}{1-v}$ for some v , as in that case we obtain

$$\Pi(\alpha, \beta) = \frac{(1-u)(1-av)}{(1-a)(1-au)(1-v)} \stackrel{\text{up to constant}}{=} \frac{1-av}{(1-a)(1-au)}.$$

Table 2 summarizes the results.

Recall now that in proposition (3) we identified the local factor and immediately afterward, we saw that the dimension reduction allows us, when k hyperbolic planes have been taken away and $B^m = B^{n-2k}$ is anisotropic, to draw a connection between Π^n and Π^m . We conclude that, when $\beta = 0$, the local factor is

$$\frac{\Pi^m(\alpha - n, k)}{q^{-\alpha} Z(\alpha)}$$

up to a multiplicative constant.

Finally, equations (28) and (29) give us the (uncorrected) global period. Recall that when $\Delta = 1$ we use $\chi = \chi_0$ and the local factor of $L(\cdot, \chi) = \zeta(\cdot)$ is $Z(\cdot)$, while when $\Delta = -1$ we use $\chi = \chi_1$ and the local factor of $L(\cdot, \chi)$ is 1. Table 3 summarizes the conclusions.

REFERENCES

- [1] Avraham Aizenbud, Dmitry Gourevitch, and Eitan Sayag, *$(O(V \oplus F), O(V))$ is a Gelfand pair for any quadratic space V over a local field F* , Math. Z. **261** (2009), no. 2, 239–244. MR2457297 (2010a:22020), Zbl 1179.22017
- [2] João Pedro Boavida, *Compact periods of Eisenstein series of orthogonal groups of rank one*, Indiana U. Math. J. **62** (2013), no. 3, 869–890. MR3164848, Zbl 1301.11049

TABLE 2. Taking k hyperbolic planes from B^n , we obtain an anisotropic form B^m . With $z = q^{-k}$, $w = q^{-k-1}$, and $u = q^{-n} = q^{-m-2k} = z^2 q^{-m}$, we obtain $\Pi^m(al, k)$ from $X^m(k; t^2)$. When relevant, we indicate the choice of v that permits the simplification discussed in the text. We drop any common factors that do not depend on a .

n	k	m	v	$\Pi^m(\alpha, k)$
$3 + 8\ell$	4ℓ	3		$\frac{1}{(1-a)(1-au)} = Z(\alpha) Z(\alpha + n);$
$4 + 8\ell$	$4\ell + 1$	2	$-w$	$\frac{1+aw}{(1-a)(1-au)} = \frac{Z(\alpha) Z(\alpha + n) Z(\alpha + \frac{n}{2})}{Z(2\alpha + n)};$
$5 + 8\ell$	$4\ell + 2$	1	$-z$	$\frac{1+az}{(1-a)(1-au)} = \frac{Z(\alpha) Z(\alpha + n) Z(\alpha + k)}{Z(2\alpha + n - 1)};$
$6 + 8\ell$	$4\ell + 3$	0		1;
$7 + 8\ell$	$4\ell + 3$	1	$\frac{2u}{1+w+u}$	$\frac{1-av}{(1-a)(1-au)} = \frac{Z(\alpha) Z(\alpha + n)}{Z(\alpha - \log_q v)};$
$8 + 8\ell$	$4\ell + 3$	2	w	$\frac{1-aw}{(1-a)(1-au)} = \frac{Z(\alpha) Z(\alpha + n)}{Z(\alpha + k + 1)};$
$9 + 8\ell$	$4\ell + 3$	3	w	$\frac{1-aw}{(1-a)(1-au)} = \frac{Z(\alpha) Z(\alpha + n)}{Z(\alpha + k + 1)};$
$10 + 8\ell$	$4\ell + 3$	4		$\frac{(1-aw)(1+awq^{-1})}{(1-a)(1-au)} = \frac{Z(\alpha) Z(\alpha + n) Z(\alpha + \frac{n}{2})}{Z(\alpha + k + 1) Z(2\alpha + n)}.$

- [3] ———, *A spectral identity for second moments of Eisenstein series of $O(n, 1)$* , Illinois J. Math. **57** (2013), no. 4, 1111–1130. MR3285869, Zbl1302.11031
- [4] J. W. S. Cassels, *Rational quadratic forms*, London Mathematical Society Monographs, vol. 13, Academic Press Inc., London, 1978. MR522835 (80m:10019), Zbl0395.10029
- [5] Adrian Diaconu and Paul Garrett, *Integral moments of automorphic L -functions*, J. Inst. Math. Jussieu **8** (2009), no. 2, 335–382. MR2485795 (2010h:11081), Zbl1268.11065
- [6] ———, *Subconvexity bounds for automorphic L -functions*, J. Inst. Math. Jussieu **9** (2010), no. 1, 95–124. MR2576799 (2011b:11064), Zbl1275.11084
- [7] ———, *Averages of symmetric square L -functions, and applications* (2009), preprint, available at http://www.math.umn.edu/~garrett/m/v/sym_two.pdf.
- [8] Adrian Diaconu, Paul Garrett, and Dorian Goldfeld, *Moments for L -functions for $GL_r \times GL_{r-1}$* , Patterson 60++ International Conference on the Occasion of the 60th Birthday of Samuel J. Patterson (University of Göttingen, July 27–29, 2009), Contributions in analytic and algebraic number theory (Valentin Blomer and Preda Mihăilescu, eds.), Springer Proc. Math., vol. 9, Springer, New York, 2012, pp. 197–227. MR3060461, Zbl06221503
- [9] Paul Garrett, *Euler factorization of global integrals*, NSF–CBMS Regional Conference in Mathematics on Euler Products and Eisenstein Series (Texas Christian University, Fort Worth, Tex., May 20–24, 1996), Automorphic forms, automorphic representations, and arithmetic (Robert S. Doran, Ze-Li Dou, and George T. Gilbert, eds.), Proc. Sympos. Pure Math., vol. 66.2, Amer. Math. Soc., Providence, R.I., 1999, pp. 35–101. MR1703758 (2000m:11043), Zbl1002.11042
- [10] Benedict H. Gross and Dipendra Prasad, *Test vectors for linear forms*, Math. Ann. **291** (1991), no. 2, 343–355. MR1129372 (92k:22028), Zbl0768.22004

TABLE 3. For each n , we list the uncorrected global period as well as the correction factor at $p = 2$, ignoring multiplicative constants independent of α .

n	uncorrected period	correction factor at $p = 2$
$3 + 8\ell$	$\frac{\zeta(\alpha - n)}{\zeta(\alpha - 4\ell - 1)}$	$\frac{Z(\alpha - 4\ell - 1)}{q^{-\alpha}};$
$4 + 8\ell$	$\frac{\zeta(\alpha - n) L(\alpha - \frac{n}{2}, \chi)}{\zeta(2\alpha - n)}$	$\frac{Z(\alpha - \frac{n}{2})}{q^{-\alpha}};$
$5 + 8\ell$	$\frac{\zeta(\alpha - n)}{L(\alpha - 4\ell - 2, \chi)}$	$\frac{Z(\alpha - 4\ell - 3)}{Z(2\alpha - n - 1) q^{-\alpha}};$
$6 + 8\ell$	$\frac{\zeta(\alpha - n) \zeta(\alpha - \frac{n}{2})}{\zeta(2\alpha - n)}$	$\frac{Z(2\alpha - n)}{Z(\alpha) Z(\alpha - n) Z(\alpha - \frac{n}{2}) q^{-\alpha}};$
$7 + 8\ell$	$\frac{\zeta(\alpha - n)}{\zeta(\alpha - 4\ell - 1)}$	$\frac{Z(\alpha - 4\ell - 1)}{Z(\alpha - n - 1 - \log_q(1 + q^{-4\ell-4} + q^{-n})) q^{-\alpha}};$
$8 + 8\ell$	$\frac{\zeta(\alpha - n) L(\alpha - \frac{n}{2}, \chi)}{\zeta(2\alpha - n)}$	$\frac{Z(2\alpha - n)}{Z(\alpha - \frac{n}{2}) q^{-\alpha}};$
$9 + 8\ell$	$\frac{\zeta(\alpha - n)}{L(\alpha - 4\ell - 2, \chi)}$	$\frac{1}{Z(\alpha - 4\ell - 5) q^{-\alpha}};$
$10 + 8\ell$	$\frac{\zeta(\alpha - n) \zeta(\alpha - \frac{n}{2})}{\zeta(2\alpha - n)}$	$\frac{1}{Z(\alpha - 4\ell - 6) q^{-\alpha}}.$

- [11] ———, *On the decomposition of a representation of SO_n when restricted to SO_{n-1}* , *Canad. J. Math.* **44** (1992), no. 5, 974–1002. MR1186476 (93j:22031), Zbl0787.22018
- [12] ———, *On irreducible representations of $SO_{2n+1} \times SO_{2m}$* , *Canad. J. Math.* **46** (1994), no. 5, 930–950. MR1295124 (96c:22028), Zbl0829.22031
- [13] Benedict H. Gross and Mark Reeder, *From Laplace to Langlands via representations of orthogonal groups*, *Bull. Amer. Math. Soc. (N.S.)* **43** (2006), no. 2, 163–205. MR2216109 (2007a:11159), Zbl1159.11047
- [14] Atsushi Ichino and Tamotsu Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross–Prasad conjecture*, *Geom. Funct. Anal.* **19** (2010), no. 5, 1378–1425. MR2585578, Zbl1216.11057
- [15] Henryk Iwaniec and Peter Sarnak, *Perspectives on the analytic theory of L -functions*, *Visions in Mathematics, towards 2000* (Tel Aviv University, August 25–September 3, 1999), GAFA 2000 (N. Alon, J. Bourgain, A. Connes, M. Gromov, and V. Milman, eds.), Birkhäuser, Basel, 2000, pp. 705–741. *Geom. Funct. Anal.*, Special Volume, Part II. MR1826269 (2002b:11117), Zbl0996.11036
- [16] Shin-ichi Kato, Atsushi Murase, and Takashi Sugano, *Whittaker–Shintani functions for orthogonal groups*, *Tohoku Math. J. (2)* **55** (2003), no. 1, 1–64. MR1956080 (2003m:22020), Zbl1037.22034
- [17] Hervé Jacquet, Erez Lapid, and Jonathan Rogawski, *Periods of automorphic forms*, *J. Amer. Math. Soc.* **12** (1999), no. 1, 173–240. MR1625060 (99c:11056), Zbl1012.11044
- [18] Dihua Jiang, *Periods of automorphic forms*, *Proceedings of the International Conference on Complex Geometry and Related Fields* (Stephen S.-T. Yau, Zhijie Chen, Jianpan Wang, and Sheng-Li Ten, eds.), *AMS/IP Stud. Adv. Math.*, vol. 39, Amer. Math. Soc., Providence, R.I., 2007, pp. 125–148. MR2338623 (2008k:11052), Zbl1161.11009
- [19] Erez Lapid and Omer Offen, *Compact unitary periods*, *Compos. Math.* **143** (2007), no. 2, 323–338. MR2309989 (2008g:11091), Zbl1228.11073

- [20] Erez Lapid and Jonathan Rogawski, *Periods of Eisenstein series*, C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), no. 6, 513–516. MR1860921 (2002k:11072), Zbl1067.11028
- [21] Delia Letang, *Automorphic spectral identities and applications to automorphic L -functions on GL_2* , J. Number Theory **133** (2013), no. 1, 278–317. MR2981412, Zbl06110193
- [22] Atsushi Murase and Takashi Sugano, *Shintani function and its application to automorphic L -functions for classical groups. I. The case of orthogonal groups*, Math. Ann. **299** (1994), no. 1, 17–56. MR1273075 (96c:11054), Zbl0813.11032
- [23] O. Timothy O’Meara, *Introduction to quadratic forms*, Classics in Mathematics, Springer-Verlag, Berlin, 2000. Reprint of the 1973 edition. MR1754311 (2000m:11032), Zbl1034.11003
- [24] Vladimir Platonov and Andrei Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press Inc., Boston, Mass., 1994. Translated from the 1991 Russian original by Rachel Rowen. MR1278263 (95b:11039), Zbl0841.20046
- [25] Yiannis Sakellaridis, *Spherical varieties and integral representations of L -functions*, Algebra Number Theory **6** (2012), no. 4, 611–667. MR2966713, Zbl1253.11059
- [26] Yiannis Sakellaridis and Akshay Venkatesh, *Periods and harmonic analysis on spherical varieties* (2012), preprint, available at [arXiv:1203.0039v1](https://arxiv.org/abs/1203.0039v1) [math.RT].
- [27] André Weil, *Adeles and algebraic groups*, Progress in Mathematics, vol. 23, Birkhäuser, Boston, Mass., 1982. MR670072 (83m:10032), Zbl0493.14028

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